Computational algorithms for thermoelastic problem

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The isotropic linear elasticity problem

\[ \mu \Delta u + (\lambda + \mu) \nabla \text{div} \ u = 0, \quad x \in \Omega, \]
\[ u(x, t) = 0, \quad x \in \Gamma_D, \]
\[ (\sigma \cdot n)(x, t) = f(x, t), \quad x \in \Gamma_N, \]
\[ (\sigma \cdot n)(x, t) = 0, \quad x \in \partial \Omega \setminus \Gamma_D \setminus \Gamma_N. \]
Temperature

Temperature problem

\[ c\rho \frac{\partial T}{\partial t} - \text{div}(\lambda \text{grad} \ T) = 0, \quad x \in \Omega, \]

\[ \frac{\partial T}{\partial n} = 0, \quad x \in \Gamma_N, \]

\[ T = g(x, t), \quad x \in \Gamma_D, \]

\[ -\lambda \frac{\partial T}{\partial n} = \alpha(T - T_{out}), \quad x \in \Gamma_R. \]
Thermoelastic problem

Mathematical model

\[ \mu \nabla^2 u + (\lambda + \mu) \text{grad div } u - (3\lambda + 2\mu) \alpha_T \text{grad} (T - T_0) = 0, \]
\[ c_\varepsilon \frac{\partial T}{\partial t} + (3\lambda + 2\mu) \alpha_T T_0 \frac{\partial \text{div } u}{\partial t} = \text{div } (k \text{ grad } T). \]

Boundary conditions:

\[ \sigma(x) = g, \quad x \in \Gamma^u_N, \quad u(x) = f, \quad x \in \Gamma^u_D, \]
\[ -k \frac{\partial T}{\partial n}(x, t) = a, \quad x \in \Gamma^T_N, \quad T(x, t) = 0, \quad x \in \Gamma^T_N D \]

Initial condition:

\[ T(x, 0) = T_0, \quad x \in \Omega, \]
Approximation by time

\[
\mu \nabla^2 u + (\lambda + \mu) \text{grad} \text{div} u - \alpha \text{grad} T = 0,
\]

\[
\tilde{c} \frac{\partial T}{\partial t} + \alpha \frac{\partial \text{div} u}{\partial t} = \text{div} \left( \tilde{k} \text{grad} T \right),
\]

where \( \alpha = (3\lambda + 2\mu)\alpha_T, \tilde{c} = c_\varepsilon / T_0, \tilde{k} = / T_0. \)

Discrete-time form

\[
\mu \nabla^2 u^{n+1} + (\lambda + \mu) \text{grad} \text{div} u^{n+1} - \alpha \text{grad} T^{n+1} = 0,
\]

\[
\alpha \text{div} u^{n+1} + \tilde{c} T^{n+1} - \tau \text{div} \left( \tilde{k} \text{grad} T^{n+1} \right) = \tilde{c} T^n + \alpha \text{div} u^n.
\]
Weak formulation

Find $T \in H(\Omega)$, $u \in H(\text{div}, \Omega)$ such that

$$
\int_{\Omega} \sigma(u^{n+1}) \varepsilon(v) \, dx - \int_{\Omega} \alpha(\text{grad } T^{n+1}, v) \, dx = \int_{\Gamma_N} (g, v) \, ds,
$$

$$
\int_{\Omega} \alpha \text{div } u^{n+1} \, q \, dx + \int_{\Omega} \tilde{c} T^{n+1} \, q \, dx - \int_{\Omega} \tau(\tilde{k} \text{grad } T^{n+1}, \text{grad } q) \, dx
$$

$$
= \int_{\Omega} \tilde{c} T^n \, q \, dx + \int_{\Omega} \alpha \text{div } u^n \, q \, dx.
$$

for all $q \in H(\Omega)$, $v \in H(\text{div}, \Omega)$. 
Implementation

The process of numerical solution of the problem:

- Building geometry and mesh generation;
- Numerical implementation of the solver using Fenics library;
- Visualization of the results.
Geometry and grids

372 cells
1504 cells
5828 cells
Numerical result. Temperature

10 min

30 min

1 hour
Numerical result. Displacement
Numerical comparision of schemes with weight

**Discrete-time form**

\[ \mu \nabla^2 u^{n+1} + (\lambda + \mu) \text{grad div } u^{n+1} - \alpha \text{grad } T^{n+1} = 0, \]
\[ \alpha \text{div } u^{n+1} + \tilde{c} T^{n+1} - w \tau \text{div } (\tilde{k} \text{grad } T^{n+1}) = \]
\[ \tilde{c} T^n + \alpha \text{div } u^n + (1 - w) \tau \text{div } (\tilde{k} \text{grad } T^n). \]
Split schemes

Solution strategies:

- **Coupled schemes**
  The coupled governing equations of temperature and mechanics are solved simultaneously at every time step.

- **Split schemes**
  Either the temperature, or mechanical, problem is solved first, and then the other problem is solved using the intermediate solution information.
Split schemes

L-split (Drained split):
- freezing the temperature during solution of the mechanic problem;
- uses updated mechanics when solving temperature problem.

\[
\mu \nabla^2 u^{n+1} + (\lambda + \mu) \text{grad div } u^{n+1} - \alpha \text{grad } T^n = 0,
\]
\[
\alpha \text{div } u^{n+1} + \tilde{c} T^{n+1} - \tau \text{div } \left( \tilde{k} \text{grad } T^{n+1} \right) = \tilde{c} T^n + \alpha \text{div } u^n.
\]
Split schemes

U-split (Fixed-strain scheme):
- freezing the displacement during solution of the temperature problem;
- uses updated temperature when solving mechanic problem.

\[
\mu \nabla^2 u^{n+1} + (\lambda + \mu) \text{grad div } u^{n+1} - \alpha \text{grad } T^{n+1} = 0,
\]
\[
\alpha \text{div } u^n + \tilde{c} T^{n+1} - \tau \text{div } (\tilde{k} \text{grad } T^{n+1}) = \tilde{c} T^n + \alpha \text{div } u^{n-1}.
\]
Numerical comparison of split schemes
Numerical comparison of schemes with weight
Future works

- Poroelastic problems

\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} - \alpha \text{grad} \mathbf{p} = 0, \]
\[ \alpha \frac{\partial \text{div } \mathbf{u}}{\partial t} + S \frac{\partial \mathbf{p}}{\partial t} - \text{div} \left( \frac{k}{\mu} \text{grad } \mathbf{p} \right) = 0. \]

- Elasto-plastic problems.

Thank you for your attention!