High Accuracy Quasi interpolation of Radial Basis Functions

Ziwu Jiang

Department of Mathematics, Linyi University, Linyi 276005, China

Email: zwjiang@gmail.com
Outline

1. Radial Basis Function (RBF) Interpolation
2. Multiquadric (MQ) Quasi-interpolation
3. Construction of High Accuracy Quasi Interpolation
4. Application of High Accuracy Approximation Operator
5. Prospect of Work
RBF Interpolation

RBF was introduced by Krige in 1951 to deal with geological problems.

Definition

For a given region $\Omega \in \mathbb{R}^d$ and a finite set $X = \{x_1, \cdots, x_N\} \subset \Omega$ of distinct points, an interpolant to a given function $f$ can be constructed as

$$S_{\phi,X}(f) = \sum_{i=1}^{N} \alpha_i \phi(\|x - x_i\|_2) + \sum_{j=1}^{Q} \beta_j p_j(x), \text{ for } x \in \Omega,$$

where $\| \cdot \|_2$ denotes the Euclidean norm, $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a given RBF, and $(p_j)_{1 \leq j \leq Q}$ is a basis of the space $\mathbb{P}_m^d$ of polynomials of degree at most $m$. 
The coefficients $\alpha_i$ and $\beta_j$ in the expressing defining $S_{\phi,X}$ are determined solving the linear system

$$
\sum_{i=1}^{N} \alpha_i \phi(\|x_k - x_i\|_2) + \sum_{j=1}^{Q} \beta_j p_j(x_k) = f(x_k), \quad 1 \leq k \leq N,
$$

$$
\sum_{i=1}^{N} \alpha_i p_j(x_i) = 0, \quad 1 \leq j \leq Q.
$$

---

The most useful conditional positive definite RBFs on $\mathbb{R}^d$ are
given in the following Table:

<table>
<thead>
<tr>
<th>RBF Type</th>
<th>Formula</th>
<th>Constraint</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gaussians</strong></td>
<td>$e^{-cr^2}, \ c &gt; 0$</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td><strong>MQ-RBFs</strong></td>
<td>$(-1)^{[\mu/2]}(c^2 + r^2)^\mu, \ \mu &gt; 0, \ \mu \not\in 2\mathbb{N}_0$</td>
<td></td>
<td>$[\mu/2]$</td>
</tr>
<tr>
<td><strong>IMQ-RBFs</strong></td>
<td>$(c^2 + r^2)^\mu, \ \mu &lt; 0$</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td><strong>Thin-plate</strong></td>
<td>$(-1)^{\mu+1}r^\mu \log(r), \ \mu \in \mathbb{N}$</td>
<td></td>
<td>$[\mu/2]$</td>
</tr>
<tr>
<td><strong>Powers</strong></td>
<td>$(-1)^{[k/2]}r^k, \ k &gt; 0, \ k \not\in 2\mathbb{N}_0$</td>
<td></td>
<td>$[k/2]$</td>
</tr>
<tr>
<td><strong>CS-RBFs</strong></td>
<td>$\phi_{d,k}(r)$</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
Let
\[ \alpha = (\alpha_1, \cdots, \alpha_N)^T, \]
\[ \beta = (\beta_1, \cdots, \beta_Q)^T, \]
and
\[ f|_X = (f(x_1), \cdots, f(x_N))^T. \]

Solvability of this system is therefore equivalent to the solvability of the system
\[
\begin{pmatrix}
A_{\phi,X} & P \\
P^T & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} =
\begin{pmatrix}
f|_X \\
0
\end{pmatrix},
\]
where \( A_{\phi,X} = (\phi(\|x_j - x_k\|_2)) \in \mathbb{R}^{N \times N} \) and \( P = (p_k(x_j)) \in \mathbb{R}^{N \times Q} \).
The error estimates are expressed in terms of the fill distance

\[ h_{X,\Omega} = \sup_{x \in \Omega} \min_{x_j \in X} \| x - x_j \|_2, \]

**Theorem**

Let \( \phi \) be one of the IMQ-RBFs. Suppose that \( \Omega \) be a cube in \( \mathbb{R}^d \). Then there exists a constant \( C > 0 \) such that the error between a function \( f \in \mathcal{N}_\phi(\Omega) \) and its interpolant \( S_{f,X} \) can be bounded by

\[ \| f(x) - S_{f,X}(x) \|_{L_\infty(\Omega)} \leq e^{-C/h_{X,\Omega}} \| f \|_{\mathcal{N}_\phi(\Omega)} \]

for all data sites \( X \) with sufficiently small \( h_{X,\Omega} \).
Outline

1. Radial Basis Function (RBF) Interpolation
2. Multiquadric (MQ) Quasi-interpolation
3. Construction of High Accuracy Quasi Interpolation
4. Application of High Accuracy Approximation Operator
5. Prospect of Work
Quasi-interpolation of a univariate function $f : [a, b] \to \mathbb{R}$ with MQs at the scattered points

$$a = x_0 < x_1 < \cdots < x_n = b,$$

(2)

has the form

$$\mathcal{M}(f) := \mathcal{M}f = \sum_{i=0}^{n} f(x_i) \psi_i(x),$$

(3)

where each function $\psi_i(x)$ is a linear combination of the MQs

$$\psi_i(x) = \sqrt{c^2 + (x - x_i)^2},$$

(4)

and $c \in \mathbb{R}^+$ is a shape parameter.
where

\[ \mathcal{L}_B f(x) = \sum_{i=0}^{n} f(x_i) \tilde{\psi}_i(x), \]  

(5)

\[\begin{align*}
\tilde{\psi}_0(x) &= \frac{1}{2} + \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)}, \\
\tilde{\psi}_1(x) &= \frac{\psi_2(x) - \psi_1(x)}{2(x_2 - x_1)} - \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)}, \\
\tilde{\psi}_i(x) &= \frac{\psi_{i+1}(x) - \psi_i(x)}{2(x_{i+1} - x_i)} - \frac{\psi_i(x) - \psi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 2 \leq i \leq n - 2, \quad (6) \\
\tilde{\psi}_{n-1}(x) &= \frac{(x_n - x) - \psi_{n-1}(x)}{2(x_n - x_{n-1})} - \frac{\psi_{n-1}(x) - \psi_{n-2}(x)}{2(x_{n-1} - x_{n-2})}, \\
\tilde{\psi}_n(x) &= \frac{1}{2} + \frac{\psi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})}. 
\end{align*}\]
\( \mathcal{L}_D \) has the following properties:

**Theorem (Wu & Schaback)**

For \( f \in C^2[a, b] \) the quasi-interpolant \( \mathcal{L}_D f \) defined by (5)-(6) from the points in (2) satisfies an error estimate of type

\[
\| f - \mathcal{L}_D f \|_\infty \leq (C_1 h^2 + C_2 c h + C_3 c^2 \log h) \| f'' \|_\infty,
\]

for \( h := \max_{1 \leq i \leq n} (x_i - x_{i-1}) \) with suitable positive constants \( C_1, C_2, \) and \( C_3, \) independent of \( h \) and \( c. \)

---

Outline

1. Radial Basis Function (RBF) Interpolation
2. Multiquadric (MQ) Quasi-interpolation
3. Construction of High Accuracy Quasi Interpolation
4. Application of High Accuracy Approximation Operator
5. Prospect of Work
The procedure of the construction

\[ f - g \text{ satisfies } \| (f - g)'' \|_{\infty} < \varepsilon \]

\[ \| (f - g) - L_D (f - g) \|_{\infty} < \varepsilon \cdot h^2 \]

\[ \| f - (g + L_D (f - g)) \|_{\infty} < \varepsilon \cdot h^2 \]

\[ Lf = g + L_D (f - g) \]

\[ g = \sum_i \alpha_i \sqrt{c^2 + (\cdot - x_i)^2} \]

\[ g'' = \sum_i \alpha_i \frac{c^2}{(c^2 + (\cdot - x_i)^2)^{3/2}} \]
Let $N < n$, and $0 < k_1 < k_2 < \cdots < k_N < n$. Suppose that

$$h_2 := \max_{2 \leq i \leq N} (x_{k_i} - x_{k_{i-1}}). \quad (7)$$

we can get a RBF interpolant $S_{f''}^N$ to approximate the second derivative of $f(x)$ on $[a, b]$, and satisfying

$$S_{f''}^N(x_{k_i}) = f''(x_{k_i}), \quad i = 1, \cdots, N.$$

Here, we use the IMQ-RBF

$$\phi(r) = \frac{s^2}{(s^2 + r^2)^{3/2}},$$

which can be derived from the second derivative of MQ-RBF

$$\psi(r) = \sqrt{s^2 + r^2}, \quad s > 0.$$
IMQ-RBF is a strictly positive definite function, we can express the interpolant $S_{f''_N}$ as follow

$$S_{f''}(x) = \sum_{i=1}^{N} \alpha_i \phi(|x - x_k|),$$  \hspace{1cm} (8)

and the coefficients $\{\alpha_i\}_{i=0}^{N}$ are determined by the interpolation condition

$$S_{f''}(x_{kj}) = \sum_{i=1}^{N} \alpha_i \phi(|x_{kj} - x_{ki}|) = f''(x_{kj}), \hspace{0.5cm} j = 1, \cdots, N.$$  \hspace{1cm} (9)
By using \( f \) and the coefficient \( \alpha \), we construct a function in the form

\[
e(x) = f(x) - \sum_{i=1}^{N} \alpha_i \sqrt{s^2 + (x - x_{k_i})^2}.
\]  

(10)

Then we can define a MQ quasi-interpolation on the data \((x_i, e(x_i))_{1 \leq i \leq n}\) with the shape parameter \( c \).

Then the new MQ quasi-interpolation operator is presented as

\[
\mathcal{L}_W f(x) = \sum_{i=1}^{N} \alpha_i \sqrt{s^2 + (x - x_{k_i})^2} + \mathcal{L}_D e(x).
\]  

(11)

Generally speaking, the shape parameters \( c \) and \( s \) should not the same constant in (11).
Unfortunately, values of the function are given in practice rather than the second derivatives. So in (7), we can replace $f''(x_{kj})$ by

$$
\frac{f(x_{kj+1}) - 2f(x_{kj}) + f(x_{kj-1})}{h^2}, \quad \text{with} \quad h = \frac{b - a}{n},
$$

(12)

when the data $(x_i, f(x_i))_{0 \leq i \leq n}$ are given, and $(x_i)_{0 \leq i \leq n}$, are equally spaced points.

Let $F''_{X'} = (f''_{x_{k1}}, \ldots, f''_{x_{k1}})^T$, if we replace $f''_{X'}$ in by $F''_{X'}$, then we use $\mathcal{L}_{W_2}$ to denote the quasi-interpolation operator defined by (10), and (11).
We conclude our results on the error estimate of the quasi-interpolant $\mathcal{L}_\mathcal{M}$ with the following result.

**Theorem (Jiang&Wang etc.)**

For a given function $f$, suppose that $f'' \in \mathcal{N}_\phi$, $c^2|\log c| = \mathcal{O}(h^2)$, and $h_2 = \mathcal{O}(h)$. Then there exists constants $C_1, C_2 > 0$ such that

$$\|f - \mathcal{L}_\mathcal{M}f\|_\infty \leq C_1 h^2 e^{-C_2/h} \|f''\|_{\mathcal{N}_\phi}.$$ 

For the other MQ quasi-interpolation scheme $\mathcal{L}_{W_2}$, we have the following conclusion about its error estimate.

**Theorem**

If $(x_i)_{0 \leq i \leq n}$ are equally spaced points, function $f(x)$ satisfied that $f'''(x) \in \mathcal{N}_\phi$, and $|f^{(4)}(x)| < M$, where $M > 0$ is a constant. Suppose that $c^2|\log c| = O(h^2)$, and $s = O(h_2)$, then there are constants $C_1, C_2, C_3 > 0$ such that

$$
\|f(x) - \mathcal{L}_{W_2}f(x)\|_\infty \leq h^2 \left( C_1 e^{-C_2/h} \|f''\|_{\mathcal{N}_\phi} + C_3 h^2 / h_2^2 \right).
$$
Outline

1. Radial Basis Function (RBF) Interpolation
2. Multiquadric (MQ) Quasi-interpolation
3. Construction of High Accuracy Quasi Interpolation
4. Application of High Accuracy Approximation Operator
5. Prospect of Work
Example

We use

\[ f(x) = \sin(x) + 0.1 \sin(32x). \]

The graph of \( f(x) \) is shown in Figure 1.
The numerical results by using kinds of the MQ quasi-interpolations are shown in Figure 2 and 3. Here, we set \( t = h_2/h \), \( s = 10h_2 \), and \( c = h \).

**Figure:** Errors of approximation to \( f(x) \) by using \( \mathcal{L}_D \), \( \mathcal{L}_R \), and \( \mathcal{L}_W \).
Figure: Errors of approximation to $f_3(x)$ by using $L_D$, $L_R$, and $L_{W_2}$. 
We also used the approach to solve the nonlinear one-dimensional Sine–Gordon (SG) equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \sin(u), \quad a \leq x \leq b, \quad t \geq 0,
\]  

(13)

with the initial conditions

\[
u(x, 0) = f(x), \quad a \leq x \leq b, \quad (14)
\]

\[
\frac{\partial u}{\partial t}(x, 0) = g(x), \quad a \leq x \leq b, \quad (15)
\]

and the boundary conditions

\[
u(a, t) = h_1(t), \quad u(b, t) = h_2(t), \quad t \geq 0. \quad (16)
\]
Outline

1. Radial Basis Function (RBF) Interpolation
2. Multiquadric (MQ) Quasi-interpolation
3. Construction of High Accuracy Quasi Interpolation
4. Application of High Accuracy Approximation Operator
5. Prospect of Work
Prospect of Work

- Applying widely the operator to many fields, such as signal processing and so on.
- Utilize other radial basis function as the quasi-interpolation basis.
- Develope the quasi-interpolation of radial basis functions to high dimensional space.
Thanks!